Herringbone pattern for an anisotropic complex Swift-Hohenberg equation

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We show using numerical simulations that herringbone type traveling wave patterns arise as a dissipative structure in an anisotropic complex Swift-Hohenberg equation. In the herringbone pattern, zig and zag rolls alternate spatially along the *x* direction. The herringbone pattern becomes unstable as a control parameter is changed, then irregular patches of zig and zag structures appear. [S1063-651X(98)15312-6]

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Various types of pattern formation have been studied in many spatially extended dissipative systems [1]. Wave patterns are dissipative structures generated by the oscillatory instability setting in at a finite wavelength of small amplitude. There are two kinds of wave patterns, i.e., traveling waves and standing waves. Traveling waves were found in experiments of binary fluids [2] and in electrohydrodynamic convection of liquid crystals. The chevron patterns in liquid crystals are composed of traveling wave patterns and take herringbonelike structures in a certain parameter region [3,4]. Recently a localized traveling wave pattern and chaotic traveling wave patterns composed of zig and zag rolls were found in experiments on liquid crystals [5,6].

The complex Swift-Hohenberg equation has been studied as a model equation for traveling wave patterns as a dissipative structure [7,8]. Localized traveling wave patterns were found in numerical simulations of the complex Swift-Hohenberg equation [9,10]. In this paper, we show using numerical simulations that herringbone type traveling wave patterns and chaotic patterns arise in an anisotropic complex Swift-Hohenberg equation.

As an anisotropic model equation, we study the complex Swift-Hohenberg equation, which takes the form

$$\begin{aligned}\partial_t u=&(a+if_0)u-b\{(q_{0x}^2+\partial_{xx})^2+(q_{0y}^2+\partial_{yy})^2\}u\\&+i(f_{1x}\partial_{xx}+f_{2x}\partial_{xxxx}+f_{1y}\partial_{yy}+f_{2y}\partial_{yyyy})u\\&-d_1|u|^2u-d_2|\partial_x u|^2u-d_3|\partial_y u|^2u,\end{aligned}\tag{1}$$

where u is the complex order parameter for a wave pattern, $d_1=d_1'+id_1''$ and $d_2=d_2'+id_2''$, $d_3=d_3'+id_3''$ are complex coefficients for the nonlinear terms, and a is a control parameter. The parameters b,f_0 and $f_{1x}\sim f_{2y}$ are determined from a linear dispersion relation. Wave patterns with specific wave vectors $(\pm q_{0x},\pm q_{0y})$ appear at the instability point $a=0$. In this paper we assume $b=1/4$, $d_3=0$, $q_{0x}=q_{0y}=1$ for the sake of simplicity. If a is sufficiently small, we can express u as a linear combination of the right-traveling zig-zag modes and left-traveling zig-zag modes:

$$\begin{aligned}u=&W_{++}(x,y,t)e^{i(q_{0x}x+q_{0y}y+\omega t)}\\&+W_{-+}(x,y,t)e^{i(-q_{0x}x+q_{0y}y+\omega t)}\\&+W_{+-}(x,y,t)e^{i(q_{0x}x-q_{0y}y+\omega t)}\\&+W_{--}(x,y,t)e^{i(-q_{0x}x-q_{0y}y+\omega t)},\end{aligned}$$

where $W_{++}\sim W_{--}$ are complex amplitudes for the four traveling wave modes and $\omega=f_0-f_{1x}q_{0x}^2+f_{2x}q_{0x}^4-f_{1y}q_{0y}^2+f_{2y}q_{0y}^4$ is the frequency of the waves. If only one of the four complex amplitudes is not zero, there appears an obliquely traveling wave pattern. If all of the four complex amplitudes are not zero and the amplitudes have the same magnitude, there appears a rectangular type standing wave pattern. Which type of wave pattern arises depends on the relative magnitude of the complex coefficients $d_1\sim d_3$ for the nonlinear terms. We consider a case in which only W_{++} and W_{-+} are not zero. If a is sufficiently small, we can derive the coupled complex Ginzburg-Landau equations for W_{++} and W_{-+} as

$$\begin{aligned}\partial_t W_{++}=&aW_{++}+(-2f_{1x}+4f_{2x})\partial_x W_{++}+(-2f_{1y}+4f_{2y})\partial_y W_{++}+\{1+i(f_{1x}-6f_{2x})\}\partial_{xx}W_{++}\\&+\{1+i(f_{1y}-6f_{2y})\}\partial_{yy}W_{++}-(d_1+d_2)|W_{++}|^2W_{++}-2d_1|W_{-+}|^2W_{++},\end{aligned}$$

$$\begin{aligned}\partial_t W_{-+}=&aW_{-+}-(-2f_{1x}+4f_{2x})\partial_x W_{-+}+(-2f_{1y}+4f_{2y})\partial_y W_{-+}+\{1+i(f_{1x}-6f_{2x})\}\partial_{xx}W_{-+}\\&+\{1+i(f_{1y}-6f_{2y})\}\partial_{yy}W_{-+}-(d_1+d_2)|W_{-+}|^2W_{-+}-2d_1|W_{++}|^2W_{-+}.\end{aligned}\tag{2}$$

There are two types of spatially uniform solutions to Eq. (2). One is a zig or zag type obliquely traveling wave solution: $|W_{++}|=\sqrt{a/(d_1'+d_2')}$, $W_{-+}=0$, or $|W_{-+}|=\sqrt{a/(d_1'+d_2')}$, $W_{++}=0$, and the other is a rectangular type traveling wave solution: $|W_{++}|=|W_{-+}|=\sqrt{a/(3d_1'+d_2')}$. We studied the coupled complex Ginzburg-Landau equations for the case $d_1'<d_2'$ and $a=1$ in one dimension [11]. We have found various complicated dynamical behavior in the coupled complex Ginzburg-Landau equations. In a parameter range, the spatially uniform solution, which satisfies $|W_{++}|=|W_{-+}|\neq 0$, loses its stability and then a spatially periodic structure appears. In the spatially periodic structure, W_{++}-dominant regions and W_{-+}-dominant regions alternate in space. The amplitude $|W_{++}|$ (or $|W_{-+}|$) takes a form similar to the solitonlike solution for the single complex Ginzburg-Landau equation in each W_{++}-(or W_{-+}-) dominant region.

We show first some numerical results for the one-dimensional version of Eq. (1),

$$\partial_t u=(a+if_0)u-b(q_{0x}^2+\partial_{xx})^2u+i(f_{1x}\partial_{xx}+f_{2x}\partial_{xxxx})u$$
$$-d_1|u|^2u-d_2|\partial_x u|^2u, \tag{3}$$

which is obtained by omitting the terms including ∂_y in Eq. (1). We have used a pseudospectral method with 512 modes. The system size L is 64π, the time step Δt is 0.005 and periodic boundary conditions are implemented. We have taken $u(x,0)=0.05\exp(iq_{0x}x)+$ random noise as an initial condition. Figure 1 shows some results of numerical simulation for Eq. (3). The parameters are $d_1=0.45-1.6i$, $d_2=0.55+2.2i$, $q_{0x}=1$, $f_{1x}=1.5$, $f_{2x}=0.75$ and the control parameter a is changed. The parameters are in the parameter region where the spatially periodic pattern has appeared in the coupled complex Ginzburg-Landau equations (2). Figure 1(a) displays a time evolution of $\mathrm{Re}\,u$ after an initial transient time $T=1500$ at $a=0.01$. Initially a traveling wave pattern $u\sim\exp(q_{0x}x+\omega t)$ appears, but the amplitude of the inversely traveling wave mode grows since $d_1'<d_2'$. A regular standing wave state is also unstable for the parameters. After an initial transient time, a spatially periodic structure appears, in which left-traveling and right-traveling wave patterns alternate in space. In other words, sinks and sources of the inversely traveling waves are generated periodically in space. The interval between a sink and the neighboring source is $L/4$ at $a=0.01$. Figure 1(b) displays a time evolution of $\mathrm{Re}\,u$ at $a=0.04$. Similar spatially periodic structure appears but the interval between a sink and the neighboring source is $L/8$. If the coupled complex Ginzburg-Landau equations are a good approximation to the complex Swift-Hohenberg equation for sufficiently small a and the group velocity term in proportion to $2f_{1x}-4f_{2x}$ can be neglected, Eq. (2) can be transformed to a normal form with $a=1$ by a scale change: $x\rightarrow\sqrt{a}x$, $t\rightarrow at$, $W\rightarrow\sqrt{a}W$. Therefore, the interval between the neighboring sink and source is expected to decrease as $1/\sqrt{a}$, as a is increased. As a is further increased, the interval is decreased to the same order as the wavelength of the traveling waves and then the approximation by the coupled complex Ginzburg-Landau equation is not good. Our numerical simulation shows that the spatially periodic structure becomes unstable for $a>0.09$ and a chaotic pattern appears. Figure 1(c) displays a time evolution of $\mathrm{Re}\,u$ at $a=0.1$. The whole space is roughly separated into two regions. One is a fairly regular region (roughly $x<50$ and $x>150$) where left- and right-traveling wave patterns and a sink between the inversely traveling wave patterns are seen. In the other region ($50<x<150$), irregular motion appears. The irregular pattern becomes dominant as a is increased.

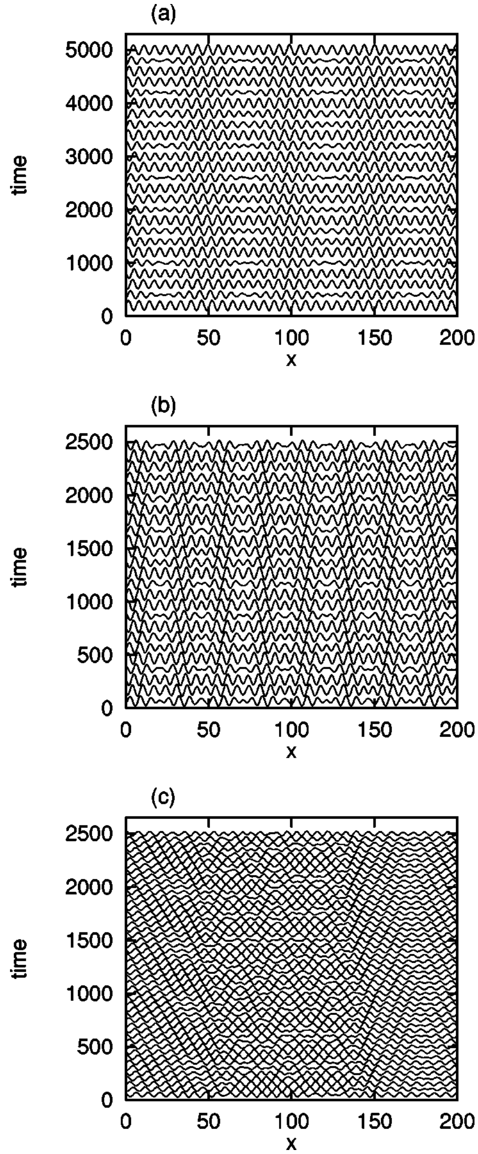

FIG. 1. Time evolution of $\mathrm{Re}\,u$ for the one-dimensional complex Swift-Hohenberg equation at (a) $a=0.01$, (b) $a=0.04$, and (c) $a=0.1$ for $d_1=0.45-1.6i$, $d_2=0.55+2.2i$, $q_{0x}=1$, $f_{1x}=1.5$, $f_{2x}=0.75$. The time in the ordinate indicates the original time t minus the initial transient time T.

We have performed numerical simulation in two dimensions for the parameters $d_1=0.45-1.6i$, $d_2=0.55+2.2i$, $q_{0x}=q_{0y}=1$, $f_{1x}=1.5$, $f_{2x}=0.75$, $f_{1y}=-0.6$, and $f_{2y}=-0.3$. The parameter values of f_{1y} and f_{2y} do not induce instability along the y direction in themselves. The system size is $L_x\times L_y=64\pi\times 16\pi$ and the pseudospectral method is used for the numerical simulations. We have taken $u=0.05\exp(iq_{0x}x+iq_{0y}y)+$random noise as an initial condition. Figure 2(a) displays a snapshot pattern of $\mathrm{Re}\,u$ at a

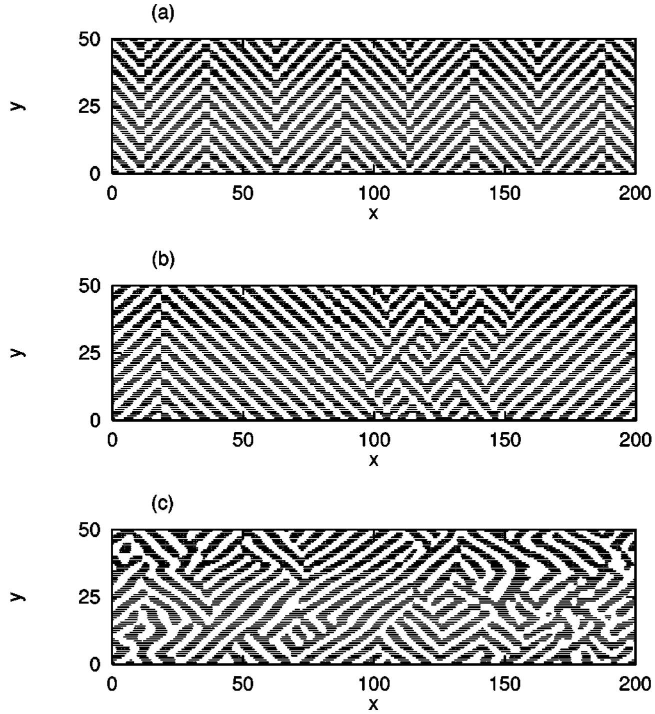

FIG. 2. Snapshot patterns of Re u for the two-dimensional complex Swift-Hohenberg equation at (a) $a=0.04$, (b) $a=0.1$, and (c) $a=0.8$ for $f_{1y}=-0.6$, $f_{2y}=-0.3$, $q_{0y}=1$. The other parameters are the same as for Fig. 1. In the shaded region, Re $u>0$.

$=0.04$. A herringbone pattern appears after an initial transient time. The herringbone pattern is composed of zig and zag type traveling wave patterns. There exist sink and source lines between the zig and zag patterns. The traveling waves are emitted from the source lines and absorbed into the sink lines. The sink and source lines do not move at the parameter. Figure 2(b) displays a snapshot pattern of Re u at $a=0.1$. There appear two regions similarly to the one dimensional case of Fig. 1(c). One is a fairly regular region that is composed of zig and zag traveling wave patterns and a straight boundary line between the zig and zag patterns. On the other hand, the boundaries between the zig and zag patterns are curved in the other irregular region. The time evolution is chaotic in the latter region, and the chaotic motion induces inhomogeneity in the y direction and makes the boundary lines curved. The transition from a one-dimensional chaotic pattern, in which the pattern is homogeneous along the y direction, to a two-dimensional chaotic pattern was studied also for another anisotropic model equation [12]. As a is further increased, the fairly regular region disappears and the whole region is composed of irregular patches of zig and zag patterns. Figure 2(c) displays such an irregular pattern of Re u at $a=0.8$.

To summarize, we have performed one- and two-dimensional simulations of the anisotropic complex Swift-Hohenberg equation. The complex Swift-Hohenberg equation (1) has a large number of parameters and is expected to exhibit various dynamical behavior depending on the parameters. We have studied a parameter region where both the two types of spatially uniform solutions: $|W_{++}|\neq 0$, $|W_{-+}|=0$ ($|W_{++}|=0$, $|W_{-+}|\neq 0$) and $|W_{++}|=|W_{-+}|\neq 0$ are unstable. We have found a spatially periodic pattern composed of inversely traveling waves in one dimension. The spatially periodic structure arises as a herringbone pattern in two dimensions. As a control parameter a is increased, more irregular patterns appear. The herringbone patterns and the irregular zig-zag patterns may have some relation to the complex traveling wave patterns found in experiments of liquid crystals [3–6].

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